Nonlinear Limit Motions of a Slightly Asymmetric Re-entry Vehicle

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Although the linear damping moment on a symmetric missile is the same for planar motion as for coning motion, nonlinear damping for these different motions is not necessarily the same. The general cubic nonlinear damping expression requires the use of two coefficients and such damping can cause either circular or planar limit motion. The addition of configurational asymmetry introduces the possibility of more complicated limit motion. The existence of one-, two-, and three-mode *stable* limit motions is studied by quasi-linear analysis and the results verified by numerical integration of the exact differential equation.

Nomenclature

a	= (cubic planar damping/cubic circular damping) −1
C_{M_d}	= aerodynamic damping moment coefficient
C_{M_α}	= static moment coefficient
$C_{\tilde{n}}, C_{\tilde{m}}$	= $\tilde{Y}$ and $\tilde{Z}$ components of aerodynamic moment divided by $(1/2)\rho V^2 S\ell$
d_0, d_2	= coefficients in the expression for C_{M_d}, Eqs. (12-14)
F	= amplitude of the forcing function
H_j	= $(\rho S\ell^3/2I_y)d_j$; $j=0, 2$
h	= linear growth rate of a small amplitude motion, Eq. (1)
h_0	= $H_0/(-M)^{1/2}$
I_x	= axial moment of inertia
I_y	= transverse moment of inertia
k_1, k_2, k_3	= magnitudes of the three modal arms
ℓ	= reference length (usually missile diameter)
M	= $(\rho S\ell^3/2I_y)\, C_{M_\alpha}$
P	= $I_x\phi'/I_y$
$\hat{P}$	= $P/(-M)^{1/2}$
R	= amplitude of the forced response, Eq. (6)
R_0	= amplitude of the natural response, Eq. (6)
r_0	= amplitude of the oscillation, Eq. (2)
S	= reference area (usually $\pi\ell^2/4$)
s	= dimensionless arclength along the missile's trajectory
t	= time, Eq. (1)
$\hat{t}$	= $\omega_0 t$
$u, \tilde{v}, \tilde{w}$	= velocity vector components in an aeroballistic nonrolling system
V	= magnitude of the velocity vector
$X, \tilde{Y}, \tilde{Z}$	= axes of the aeroballistic nonrolling system
x	= voltage, Eq. (1)
$\hat{x}$	= x/r_0
δ	= $\lvert\tilde{\xi}\rvert$, the sine of the angle between the missile's axis and the velocity vector
δ_c	= circular limit motion radius
δ_T	= trim angle produced at zero spin rate by a small, constant magnitude asymmetry moment
$\eta_1, \eta_2, \eta_3, \eta_4$	= small perturbations of (k_1, k_2, k_3, ϕ_{30}) about a set of limit motion values
θ	= orientation angle of the complex yaw $\tilde{\xi}$
λ_1, λ_2	= damping rates of the 1- and 2-arms
$\tilde{\xi}$	= complex yaw, Eq. (10)
$\hat{\xi}$	= $\tilde{\xi}/\delta_c$
ρ	= air density
ϕ	= roll angle
ϕ_1, ϕ_2	= orientation angles of the 1- and 2-arms
ϕ_{30}	= orientation angle of the 3-arm at zero roll angle
$\dot{\phi}$	= roll rate, the frequency of the forcing function
ω_0	= frequency of the oscillation, Eq. (2)

Superscripts

$(\dot{\ })$	= 1) differentiation with respect to $\hat{t}$ (Eqs. 4-9) 2) differentiation with respect to $(-M)^{1/2} s$ [Eq. (16), on]
$(\ )'$	= differentiation with respect to s
$(\bar{\ })$	= limit motion value

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I. Introduction

It is well known that a symmetric missile with linear aerodynamics has the same damping moment in planar motion as in coning motion. More precisely, the linear aerodynamic damping is not affected by the angle between the plane of the total angle of attack and the transverse angular velocity vector. Thus we can say the linear "in-plane" and "out-of-plane" damping moments are equal. Quasi-linear analysis,[1] using a cubic damping moment which retains this equality, showed that only *planar* limit motions are stable and, more generally, that any damping moment which retains this equality can have stable planar limit motions but cannot have stable circular (coning) motions.

In 1959, however, L.C. MacAllister[2] observed an oval, almost circular, limit angular motion performed by ballistic range models of re-entry configurations primarily at transonic and high subsonic speeds. The introduction of cubic damping nonlinearities that neglected this equality allowed the existence of stable *circular* limit motions[3,4] and thereby provided a theoretical basis for MacAllister's observations. This limit motion occurs when the linear damping is destabilizing, the coning cubic damping is stabilizing, and the planar cubic damping coefficient is at least twice as large as the coning cubic damping coefficient. A pertubation analysis[5] showed that these cubic damping nonlinearities plus a strongly nonlinear static moment could cause an almost circular stable limit motion. More recently, several authors[6-9] have made a variety of wind tunnel measurements of out-of-plane damping and have shown that it can be quite different from in-plane damping for cones at supersonic and hypersonic speeds.

Thesee studies applied mainly to symmetric vehicles; however, most re-entry vehicles are not exactly symmetrical, due to ablation. This slight asymmetry usually can be accounted for by assuming a vehicle-fixed constant force and moment that cause an aerodynamic trim angle. Thus, a linear stable re-entry vehicle will have a limit coning motion with a frequency equal to its spin.[10] The magnitude of the coning motion is a function of the spin and is a maximum when the spin equals its pitch rate (resonance). It is the purpose of this paper to study the more complicated limit motions performed by a slightly asymmetric re-entry vehicle acted on by cubic planar and coning damping moments.

The one-degree-of-freedom case is the forced van der Pol oscillator, which has been studied by many authors[11,12] and is elegantly summarized in a single figure by Clauser.[13] The fourth-order differential equation appropriate to the forced angular motion of a slightly asymmetric missile is much simpler than the general fourth-order equation and has been treated by an extended quasi-linear analysis.[14] We will first briefly review the results for the forced van der Pol oscillator, discuss the two cubic damping moments under consideration and the corresponding motion for a symmetric missile, and give the complete equations for a rolling asymmetric missile and their quasi-linear solution.[14] Next, the conditions for the existence and stability of one-, two-, and three-mode limit motions will be derived and their implications discussed. Finally, the validity of the work will be established by numerical integrations of the equations of motion.

II. Forced van der Pol Oscillator

A self-excited oscillator has three parameters that describe its behavior: r_0 and ω_0, the amplitude and frequency of its oscillation, and h, the linear growth rate of small amplitude motion. For these parameters, the equation for the van der Pol oscillator is

$$d^2x/dt^2 - h\,\omega_0[1-(4x^2/r_0^2)]dx/dt + \omega_0^2 x = 0 \qquad (1)$$

where the limit oscillation has the form

$$x = r_0 \sin(\omega_0 t + \Omega_0), \quad \Omega_0 \text{ arbitrary} \qquad (2)$$

Equation (1) can be simplified by rescaling the dependent and independent variables:

$$\hat{x} = x/r_0, \quad \hat{t} = \omega_0 t \qquad (3)$$

$$\ddot{\hat{x}} - h\dot{\hat{x}}(1-4\hat{x}^2) + \hat{x} = 0 \qquad (4)$$

where dots denote derivatives with respect to $\hat{t}$.

The forced van der Pol oscillator is described by an equation of the form

$$\ddot{\hat{x}} - h\dot{\hat{x}}(1-4\hat{x}^2) + \hat{x} = F\cos(\dot{\phi}\hat{t}) \qquad (5)$$

The usual analysis assumes a solution of the form

$$\hat{x} = R\sin(\dot{\phi}\hat{t} + \Omega) + R_0\sin(\hat{t} + \Omega_0) \qquad (6)$$

and derives the following relations for R, the amplitude of the forced response, and R_0, the amplitude of the natural response:

$$R_0[2R^2 + R_0^2 - 1] = 0 \qquad (7)$$

$$[(1-\dot{\phi}^2)^2 + (h\dot{\phi})^2(1-R^2-2R_0^2)^2]\,R^2 = F^2 \qquad (8)$$

For moderate growth rates ($h \sim 0.1$), $(1-\dot{\phi}^2)/h\dot{\phi}$ less than 2 means that the forcing frequency $\dot{\phi}$ is within 10% of resonance.

A summary of the implications of Eqs. (7) and (8) is given by Fig. 1, which is taken from Fig. 28 of Ref. 13. Stable pure mode responses ($R_0=0$) can only occur for $R^2>0.5$; these responses are plotted in the upper part of Fig. 1a. The upper ellipse in Fig. 1a, the locus of vertical tangents for the

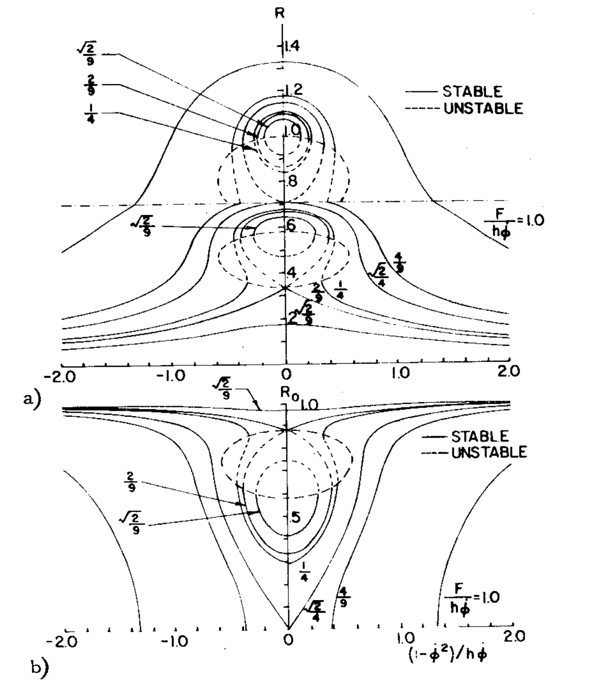

Fig. 1 a) amplitude of forced response; b) amplitude of natural response. (This figure taken from Fig. 28 of Ref. 13.)

frequency response curves, is a stability boundary and satisfies the equation

$$(1-\dot{\phi}^2)^2 + (h\dot{\phi})^2(1-R^2)(1-3R^2) = 0 \qquad (9)$$

Two-mode responses ($R_0 \neq 0$) can only occur for $R^2<0.5$ and are given in the lower part of Fig. 1a and in Fig. 1b. Note that for $F/h\dot{\phi}$ greater than $\sqrt{2}/4$, two-mode motions are not possible for intervals about resonance. Thus, only a pure mode response can occur near resonance if $F/h\dot{\phi} > \sqrt{2}/4$. For larger values of $F/h\dot{\phi}$, fairly large intervals of the forcing frequency exist for which the response locks in completely to the forcing frequency. For $F=h=0.1$, this interval extends about 7% on either side of resonance and, of course, is larger for larger values of $F/h\dot{\phi}$.

Figure 1 shows that for $F/h\dot{\phi}=4/9$ there are frequencies for which both a stable pure mode oscillation and a stable mixed mode motion are possible. Indeed, for $F/h\dot{\phi}=1/4$, one stable pure mode oscillation and two stable mixed motions are possible. Thus weakly forced systems can have a number of limit motions.

III. Nonlinear Damping Moments

The usual linear aerodynamic moment expression for a symmetric, slowly spinning missile is given in terms of the complex angle of attack and its derivative in a nonrolling coordinate system. This quantity, ξ, has a magnitude δ which is the sine of the angle between the missile's axis and the velocity vector. If the aeroballistic nonrolling X, $\tilde{Y}$, $\tilde{Z}$ axes are used, ξ is defined in terms of the transverse components of the velocity vector

$$\xi = (\tilde{v} + i\tilde{w})/V = \delta\, e^{i\theta} \qquad (10)$$

The linear aerodynamic moment coefficient then becomes

$$\begin{aligned} C_{\tilde{n}} + iC_{\tilde{m}} &= -i[C_{M_\alpha}\xi + d_0\xi'] \\ &= -i[C_{M_\alpha}\delta + d_0(\delta' + i\theta'\delta)]\, e^{i\theta} \end{aligned} \qquad (11)$$

where primes denote differentiation with respect to a dimensionless arc-length. The second form of the d_0 term—the aerodynamic damping term—shows that the damping moment in the plane of the total angle of attack is proportional to the radial rate of change of this angle and that the damping moment normal to this plane is proportional to the circumferential rate of change of this angle.

The first nonlinear extension of this form of the damping moment coefficient for a symmetric missile was written in the form

$$C_{M_d} = -i[d_0(\delta' + i\theta'\delta) + d_2\delta^2(\delta' + i\theta'\delta)] \; e^{i\theta} \qquad (12)$$

For constant d_2, Eq. (12) gives a cubic damping term that retains the in-plane and out-of-plane damping equality of the linear expansion of Eq. (11). A further generalization is to make d_2 a function of δ^2 since symmetry implies that d_2 should be an *even* function of δ. Even this generalization is not sufficient, however, to generate the almost circular limit motions observed by MacAllister.[2]

A successful approach is to drop the equality of in-plane and out-of-plane damping:

$$C_{M_d} = -i\left[d_0(\delta' + i\theta'\delta) + d_2\delta^2[(1+a)\delta' + i\theta'\delta]\right]e^{i\theta} \quad (13)$$

For constant d_2 and a, Eq. (13) yields two cubic damping terms. (The equation can, of course, be generalized by letting d_2 and a be functions of δ^2.) This more complicated nonlinear damping expression still retains the rotational symmetry of the linear expansion of Eq. (11), as can be seen by replacing the linear damping coefficient by C_{M_d} from Eq. (13) and expressing the result in terms of $\tilde{\xi}$ and $\tilde{\xi}'$:

$$C_{\tilde{n}} + iC_{\tilde{m}} = -i[(C_{M_\alpha} + d_2 a\delta\,\delta')\;\tilde{\xi} + (d_0 + d_2\delta^2)\;\tilde{\xi}'] \quad (14)$$

It should be noted that the two terms involving d_2 in Eq. (14) appear in the Maple-Synge[15] expansion of the force and moment for a body of revolution. In order to identify the corresponding Maple-Synge terms, δ^2 and $(\delta^2)'$ must be replaced by $\tilde{\xi}\bar{\tilde{\xi}}$ and $(\tilde{\xi}\bar{\tilde{\xi}})'$.

The moment coefficient of Eq. (14) yields the following differential equation for the pitching and yawing of a symmetric missile:

$$\tilde{\xi}'' + (H_0 + H_2\delta^2 - iP)\;\tilde{\xi}' - (M - H_2 a\;\delta\;\delta')\;\tilde{\xi} = 0 \qquad (15)$$

The quasi-linear analysis reveals that a circular singularity exists at $\delta^2{}_c = -H_0/H_2$ if H_0 and H_2 are opposite in sign. Considering only this case where a circular singularity can exist, we can rescale our angles with respect to δ_c and our independent variable with respect to the missile's nonrolling natural frequency $\sqrt{-M}$:

$$\ddot{\hat{\xi}} + h_0[(1-\hat{\delta}^2)\;\dot{\hat{\xi}} - a\hat{\delta}\dot{\hat{\delta}}\;\hat{\xi}] - i\hat{P}\dot{\hat{\xi}} + \hat{\xi} = 0 \qquad (16)$$

For slow spin† ($\hat{P} \sim 0$) and planar motion ($\theta = $ const), Eq. (16) reduces to van der Pol's Eq. (4) with $a = 0$, $h_0 = -h$, $\hat{\delta} = 2\hat{x}$.

The quasi-linear analysis[1] assumes an epicycle solution

$$\hat{\xi} = k_1 e^{i\phi_1} + k_2 e^{i\phi_2} \qquad (17)$$

where $\dot{k}_j = \lambda_j k_j$; $\dot{\phi}_1 = -\dot{\phi}_2 = 1$

$$\lambda_1 = -h_0(1 - k_1^2 - ak_2^2)/2$$

$$\lambda_2 = -h_0(1 - ak_1^2 - k_2^2)/2$$

The behavior of a nonlinear solution can be described by trajectories in a k_1^2 vs k_2^2 amplitude plane. Since Eq. (17) for $\lambda_j = 0$ generates ellipses with semimajor axis $k_1 + k_2$ and semiminor axis $|k_1 - k_2|$, each point in the amplitude plane identifies an elliptical motion and the trajectory through that point describes how this elliptical motion changes under the influence of nonlinear damping. Points on the axes represent circular motion and the line $k_1^2 = k_2^2$ is the locus of planar motions. The amplitude plane for the generalized van der Pol oscillator ($a = 0$) is given by Fig. 2. The circular limit motion is unstable but there is a stable planar limit motion with amplitude 2. This motion is precisely that of the van der Pol oscillator. The circular limit motion is unstable for $|a| < 1$ but is stable for $a > 1$. Figure 3 gives the amplitude plane for $a = 2$. Numerical integrations of Eq. (16) for $a > 1$ verify that stable circular limit motions do exist. Since $1 + a$ is the ratio of cubic planar damping to cubic circular damping, we see that this ratio must exceed two before stable circular motions can exist.

†Considerable algebraic simplification results from neglecting $\hat{P}$ for the remainder of this paper. It can, of course, be retained but it usually has very little effect on limit motions.

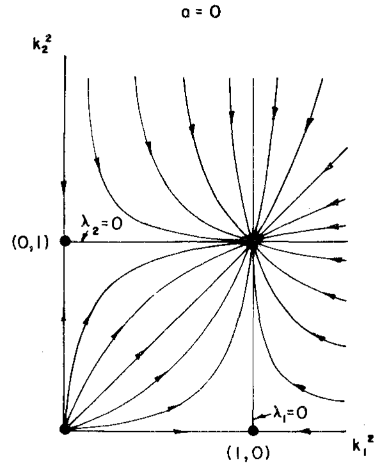

Fig. 2 Amplitude plane when the cubic planar damping equals the cubic circular damping (van der Pol oscillator).

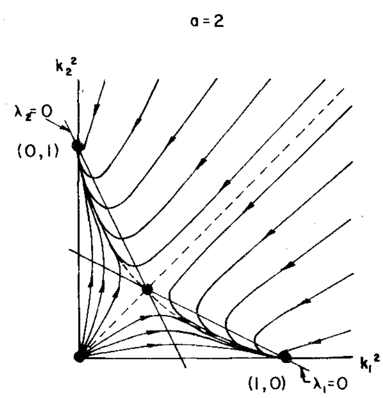

Fig. 3 Amplitude plane when the cubic planar damping is three times the cubic circular damping.

IV. Limit Motions

The effect of a small asymmetry on the moments acting on a basically symmetric missile is to add a constant magnitude moment that rotates with the missile. The size of this moment can be given in terms of the trim angle δ_T it produces at zero spin rate. The differential equation for the motion of a slightly asymmetric missile differs from Eq. (16) by the addition of an inhomogeneous term involving the roll angle:

$$\ddot{\hat{\xi}} + h_0[(1-\hat{\delta}^2)\;\dot{\hat{\xi}} \;-\; a\hat{\delta}\;\dot{\hat{\delta}}\;\hat{\xi}] + \hat{\xi} = (\delta_T/\delta_c)e^{i\phi} \qquad (18)$$

The quasi-linear analysis[14] for this equation employs a tricyclic solution:

$$\hat{\xi} = k_1 e^{i\phi_1} + k_2 e^{i\phi_2} + k_3 e^{i(\phi + \phi_{30})} \qquad (19)$$

where

$$\lambda_1 = -(h_0/2)\{1 - k_1^2 - ak_2^2 - [2 + a + \dot{\phi}\;(2-a)]k_3^2/2\} \qquad (20)$$

$$\lambda_2 = -(h_0/2)\{1 - ak_1^2 - k_2^2 - [2 + a - \dot{\phi}(2-a)]\;k_3^2/2\} \qquad (21)$$

$$\{1-\dot{\phi}^2+ih_0\ [\dot{\phi}[2-(2+a)\ (k_1^2+k_2^2)-2k_3^2]$$
$$-(2-a)\ (k_1^2-k_2^2)\ ]/2\}k_3e^{i\phi_{30}}=\delta_T/\delta_c \tag{22}$$

Three kinds of limit motions can exist: a) one-mode ($k_1=k_2=0$); b) two-mode ($k_1=\lambda_2=0$ or $k_2=\lambda_1=0$); and c) three-mode ($\lambda_1=\lambda_2=0$).

Before considering these possible limit motions in some detail, stability criteria for these motions must be derived. A particular limit motion is identified by the values of k_1, k_2, k_3, and ϕ_{30}. If we denote a set of limit motion values by $\bar{k}_1$, $\bar{k}_2$, $\bar{k}_3$, and $\bar{\phi}_{30}$, then a tricycle motion near such a limit motion is described by

$$\tilde{\xi}=(\bar{k}_1+\eta_1)e^{i\phi_1}+(\bar{k}_2+\eta_2)e^{i\phi_2}$$
$$+(\bar{k}_3+\eta_3)e^{i(\phi+\bar{\phi}_{30}+\eta_4)} \tag{23}$$

where $|\eta_j|\ll 1$.

Equation (23) can be substituted in Eq. (18) and the averaging technique of Ref. 14 used to provide linear differential equations for the η_j's. If these equations for the η_j's predict damped motion, the corresponding limit motion is stable.

V. One-Mode Motion

For one-mode motion, Eq. (22) reduces to

$$[1-\dot{\phi}^2+ih_0\dot{\phi}(1-\bar{k}_3^2)]\ \bar{k}_3\exp\ i\bar{\phi}_{30}=\delta_T/\delta_c \tag{24}$$

The angle $\bar{\phi}_{30}$ can be eliminated from Eq. (24) by squaring its real and imaginary parts and adding:

$$[(1-\dot{\phi}^2)^2+(h_0\dot{\phi})^2(1-\bar{k}_3^2)^2]\ \bar{k}_3^2=(\delta_T/\delta_c)^2 \tag{25}$$

But Eq. (25) is precisely Eq. (8) for $h_0=-h$, $F=\delta_T/\delta_c$, $R=\bar{k}_3$, and $R_0=0$. Thus, the pure mode curves of Fig. 1a are precisely those for the pure mode motion of a slightly asymmetric missile!

The perturbation equations for the pure mode limit motion are

$$\dot{\eta}_1=\bar{\lambda}_1\eta_1 \tag{26}$$

$$\dot{\eta}_2=\tilde{\bar{\lambda}}_2\eta_2 \tag{27}$$

$$\ddot{\eta}_3+h_0[1-(1+a)\bar{k}_3^2]\ \dot{\eta}_3+(1-\dot{\phi}^2)\eta_3$$
$$-2\dot{\phi}\bar{k}_3\dot{\eta}_4-h_0\dot{\phi}(1-\bar{k}_3^2)\ \bar{k}_3\eta_4=0 \tag{28}$$

$$2\dot{\phi}\dot{\eta}_3+h_0(1-3\bar{k}_3^2)\dot{\phi}\eta_3+\bar{k}_3\ddot{\eta}_4$$
$$+\bar{k}_3h_0(1-\bar{k}_3^2)\dot{\eta}_4+\bar{k}_3(1-\dot{\phi}^2)\eta_4=0 \tag{29}$$

where the $\bar{\lambda}_j$ are given by Eqs. (20) and (21) for $k_1=k_2=0$. The Routh-Hurwitz stability criteria for these differential equations are

$$-\bar{\lambda}_j>0 \tag{30a}$$
$$a_1>0 \tag{30b}$$
$$a_2a_1-\bar{k}_3a_3\equiv A>0 \tag{30c}$$
$$a_3A-a_1^2a_4>0 \tag{30d}$$
$$a_4>0 \tag{30e}$$

where the a_j's are defined in Table 1.

Table 1 Parameters in the pure-mode perturbation Eqs. (26-30)

$a_1=h_0\bar{k}_3[2-(2+a)\ \bar{k}_3^2]$
$a_2=2(1+\dot{\phi}^2)\ \bar{k}_3+h_0^2\bar{k}_3[1-(1+a)\ \bar{k}_3^2]\ (1-\bar{k}_3^2)$
$a_3\doteq h_0\bar{k}_3\{[2-(2+a)\ \bar{k}_3^2](1-\dot{\phi}^2)+4\dot{\phi}^2(1-2\bar{k}_3^2)\}$
$a_4=\bar{k}_3[(1-\dot{\phi}^2)^2+h_0^2\dot{\phi}^2(1-\bar{k}_3^2)(1-3\bar{k}_3^2)]$

For moderate damping and $\dot{\phi}$ near resonance, these inequalities can be simply stated: a) $a\leq 0$: no stable pure mode motion; b) $0<a\leq 2$: $\bar{k}_3^2>1/a$, $a_4>0$; and c) $2\leq a$: $\bar{k}_3^2>1/2$, $a_4>0$. Direct comparison of the definition of a_4 with Eq. (9) shows that $a_4>0$ is the exterior of the upper ellipse in Fig. 1a. Thus the stability predictions of Fig. 1a apply when $a\geq 2$.

As can be seen from Fig. 1a, stable pure mode solutions exist for $1<a<2$ but they are stable only for $\bar{k}_3^2\geq 1/a>1/2$. For $0<a<1$, however, stable solutions may not exist unless δ_T/δ_c is large enough. In summary then, no stable pure mode motions exist for a ≤ 0, stable pure mode motions can exist for $0<a<1$ if δ_T/δ_c is large enough, and stable pure mode motions always exist for $a\geq 1$. When these motions exist, they occur for $\dot{\phi}$ in an interval about resonant spin.

VI. Two-Mode Motion

Two possible two-mode limit motions can occur: k_2, k_3 or k_1, k_3. The conditions for k_2, k_3 limit motion ($\bar{k}_1=0$) follow from Eqs. (20-22):

$$\bar{k}_2^2=1-[2+a-(2-a)\dot{\phi}]\ \bar{k}_3^2/2 \tag{31}$$

$$\{(1-\dot{\phi}^2)^2+(h_0/4)^2\ [2(2-a-\dot{\phi})-\bar{k}_3^2[(1+\dot{\phi}^2)\ (4-a^2)$$
$$-2\dot{\phi}(2+a^2)]\ ]^2\}\ \bar{k}_3^2=(\delta_T/\delta_c)^2 \tag{32}$$

The corresponding perturbation equations for the stability determination of these two-mode limit motions are

$$\dot{\eta}_1=\bar{\lambda}_1\eta_1 \tag{33}$$

$$\dot{\eta}_2-h_0\bar{k}_2^2\eta_2+b_1\eta_3+b_2\dot{\eta}_4=0 \tag{34}$$

$$b_3\dot{\eta}_2+\ddot{\eta}_3+b_4\dot{\eta}_3+(1-\dot{\phi}^2)\eta_3-2\dot{\phi}\bar{k}_3\dot{\eta}_4+b_5\eta_4=0 \tag{35}$$

$$b_6\eta_2+2\dot{\phi}\dot{\eta}_3+b_7\eta_3+\bar{k}_3\ddot{\eta}_4+b_8\dot{\eta}_4+b_9\eta_4=0 \tag{36}$$

and the b_j's are defined in Table 2. The conditions for k_1, k_3 limit motion can be obtained by replacing $\bar{k}$ and $\dot{\phi}$ by $\bar{k}_1$ and $-\dot{\phi}$ in Eqs. (31) and (32) and by replacing

$$\bar{k}_2,\ \dot{\phi},\ \eta_1,\ \eta_2,\ \eta_4,\ \bar{\lambda}_1\ \text{by}\ \bar{k}_1,\ -\dot{\phi},\ \eta_2,\ \eta_1,\ -\eta_4,\ \bar{\lambda}_2$$

in Eqs. (33-36) and Table 2.

In order to get some feeling for the implication of these equations for two-mode limit motion, parameters were calculated from Eqs. (31) and (32) for $h_0=-0.1$, $\delta_T/\delta_c=0.1$, and six values of a. The stability of the corresponding motions were then determined by the Routh-Hurwitz criteria for perturbation equations (33-36); the results are given in Table 3.

This table shows that for these values of h_0 and δ_T/δ_c and $a>1$, stable pure mode motion occurs in an interval of spin about resonance. Outside this interval, stable two-mode limit motion of either type (k_1, k_3 or k_2, k_3) can occur. For $a<1$, stable pure mode motion cannot occur. In an interval about resonance, however, stable *two*-mode limit motion (k_2, k_3) does occur!

Table 2 Parameters in the two-mode perturbation Eqs. (33-36)

$b_1=-h_0\bar{k}_2\bar{k}_3[2+a-(2-a)\ \dot{\phi}]/2$
$b_2=h_0\bar{k}_2\bar{k}_3{}^2(2-a)/4$
$b_3=-h_0\bar{k}_2\bar{k}_3(2+3a)/2$
$b_4=h_0[2-(2+a)\ \bar{k}_2{}^2-2(1+a)\ \bar{k}_3^2]/2$
$b_5=-h_0\bar{k}_3\{\dot{\phi}[2-(2+a)\ \bar{k}_2{}^2$ $-2\bar{k}_3{}^2]+(2-a)\ \bar{k}_2{}^2\}/2$
$b_6=-h_0\bar{k}_2\bar{k}_3[(2+a)\ \dot{\phi}-2+a]$
$b_7=h_0\{\dot{\phi}[2-(2+a)\bar{k}_2{}^2$ $-6\bar{k}_3{}^2]+(2-a)\bar{k}_2{}^2\}/2$
$b_8=h_0\bar{k}_3[2-(2+a)\ \bar{k}_2{}^2-2\bar{k}_3{}^2]/2$
$b_9=(1-\dot{\phi}^2)\ \bar{k}_3$

Table 3 Stable one-, two- and three-mode limit motions

$h_0 = -0.1$; $\delta_T/\delta_c = 0.1$

	One-Mode	Two-Mode		Three-Mode
a	$\bar{k}_1 = \bar{k}_2 = 0$	$\bar{k}_1 = \lambda_2 = 0$	$\lambda_1 = \bar{k}_2 = 0$	$\lambda_1 = \lambda_2 = 0$
$-1/2$	None	$0.963 < \dot{\phi} < 1.036$	None	$\dot{\phi} < 0.926$ or $1.068 < \dot{\phi}$
0	None	$0.933 < \dot{\phi} < 1.066$	None	$\dot{\phi} < 0.914$ or $1.082 < \dot{\phi}$
$1/2$	None	$0.906 < \dot{\phi} < 1.082$	None	$\dot{\phi} < 0.902$ or $1.095 < \dot{\phi}$
$3/2$	$0.939 < \dot{\phi} < 1.057$	$\dot{\phi} < 0.910$ or $1.096 < \dot{\phi}$	$\dot{\phi} < 0.900$ or $1.099 < \dot{\phi}$	None
2	$0.931 < \dot{\phi} < 1.064$	$\dot{\phi} < 0.893$ or $1.102 < \dot{\phi}$	$\dot{\phi} < 0.893$ or $1.102 < \dot{\phi}$	None
$5/2$	$0.931 < \dot{\phi} < 1.063$	$\dot{\phi} < 0.876$ or $1.106 < \dot{\phi}$	$\dot{\phi} < 0.886$ or $1.104 < \dot{\phi}$	None

Before considering three-mode limit motions, the special case of $a=2$ should be discussed. For this case, Eqs. (31) and (32) reduce to

$$2\bar{k}_3^2 + \bar{k}_2^2 - 1 = 0 \tag{37}$$

$$\{(1-\dot{\phi}^2)^2 + (h_0\dot{\phi})^2(1-3\bar{k}_3^2)^2\}\bar{k}_3^2 = (\delta_T/\delta_c)^2 \tag{38}$$

Equations (37) and (38) are the same as Eqs. (7) and (8), and thus the remainder of Fig. 1 also applies for two-mode limit motion when $a=2$. The stability of these limit motions, however, does differ slightly from that indicated by Fig. 1.

VII. Three-Mode Motion

For three-mode limit motion, $\lambda_1 = \lambda_2 = 0$ and Eqs. (20-22) can be written as

$$\bar{k}_1^2 = \frac{1}{a+1} - \left[\frac{a+2}{a+1} + \frac{(a-2)\dot{\phi}}{a-1}\right]\bar{k}_3^2/2 \tag{39}$$

$$\bar{k}_2^2 = \frac{1}{a+1} - \left[\frac{a+2}{a+1} - \frac{(a-2)\dot{\phi}}{a-1}\right]\bar{k}_3^2/2 \tag{40}$$

$$\{(1-\dot{\phi}^2)^2 + (h_0\dot{\phi})^2[1-a+(2a^2-3)\bar{k}_3^2]^2(a^2-1)^{-2}\}\bar{k}_3^2 = (\delta_T/\delta_c)^2 \tag{41}$$

The corresponding perturbation equations are

$$\dot{\eta}_1 - h_0\bar{k}_1^2\eta_1 - a\,h_0\bar{k}_1\bar{k}_2\eta_2 + c_1\eta_3 + c_2\dot{\eta}_4 = 0 \tag{42}$$

$$-ah_0\bar{k}_1\bar{k}_2\eta_1 + \dot{\eta}_2 - h_0\bar{k}_2^2\eta_2 + c_3\eta_3 + c_4\dot{\eta}_4 = 0 \tag{43}$$

$$c_5\bar{k}_1\dot{\eta}_1 + c_5\bar{k}_2\dot{\eta}_2 + \ddot{\eta}_3 + c_6\dot{\eta}_3 + (1-\dot{\phi}^2)\eta_3 - 2\dot{\phi}\bar{k}_3\dot{\eta}_4 + c_7\eta_4 = 0 \tag{44}$$

$$c_8\eta_1 + c_9\eta_2 + 2\dot{\phi}\dot{\eta}_3 + c_{10}\eta_3 + \bar{k}_3\ddot{\eta}_4 + c_{11}\dot{\eta}_4 + c_{12}\eta_4 = 0 \tag{45}$$

where the c_j's are defined in Table 4. If the stability of three-mode limit motion is considered for the parametric values of Table 3, it develops that stable motion exists only for $a<1$. The regions of spin for which this motion is stable are given in Table 3.

Table 4 Parameters in the three-mode perturbation Eqs. (42-45)

$$c_1 = -h_0\bar{k}_1\bar{k}_3[2+a-(2-a)\dot{\phi}]/2$$
$$c_2 = -h_0\bar{k}_1\bar{k}_3^2(2-a)/4$$
$$c_3 = -h_0\bar{k}_2\bar{k}_3[2+a-(2-a)\dot{\phi}]/2$$
$$c_4 = -h_0\bar{k}_2\bar{k}_3^2(2-a)/4$$
$$c_5 = -h_0\bar{k}_3(2+3a)/2$$
$$c_6 = -h_0[2-(2-a^2)\bar{k}_3^2]/2(1+a)$$
$$c_7 = -h_0\bar{k}_3\dot{\phi}[a-1-(2a^2-3)\bar{k}_3^2]/(a^2-1)$$
$$c_8 = -h_0\bar{k}_1\bar{k}_3[(2+a)\dot{\phi}+2-a]$$
$$c_9 = -h_0\bar{k}_2\bar{k}_3[(2+a)\dot{\phi}-2+a]$$
$$c_{10} = -h_0\dot{\phi}(a-1+\bar{k}_3^2)/(a^2-1)$$
$$c_{11} = -h_0\bar{k}_3[2-(2+2a+a^2)\bar{k}_3^2]/2(1+a)$$
$$c_{12} = (1-\dot{\phi}^2)\bar{k}_3$$

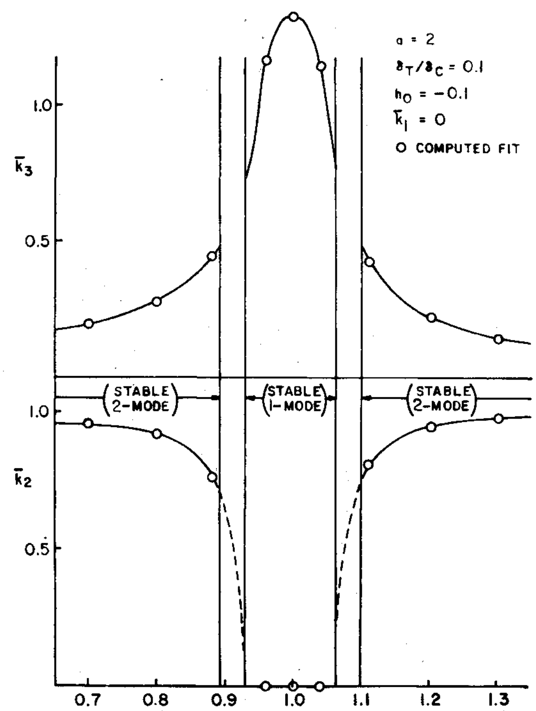

Fig. 4 For $a=2$ and $\delta_T/\delta_c = 0.1$, a comparison of Eqs. (20-22) with fits of Eq. (19) to generated solutions of Eq. (18).

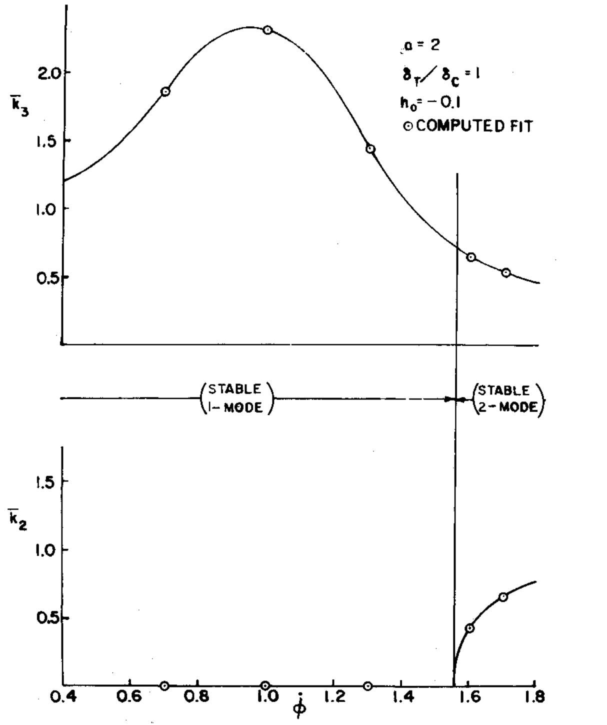

Fig. 5 For $a = 2$ and $\delta_T/\delta_c = 1$, a comparison of Eqs. (20-22) with fits of Eq. (19) to generated solutions of Eq. (18).

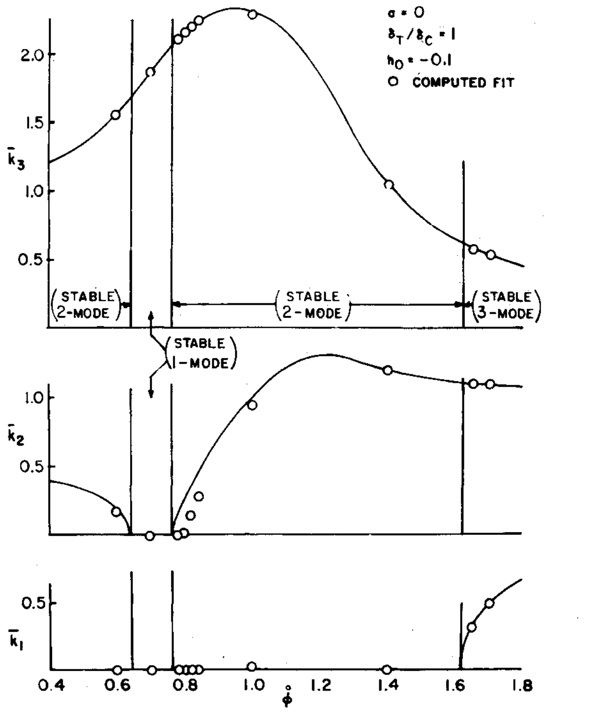

Fig. 7 For $a = 0$ and $\delta_T/\delta_c = 1$, a comparison of Eqs. (20-22) with fits of Eq. (19) to generated solutions of Eq. (18).

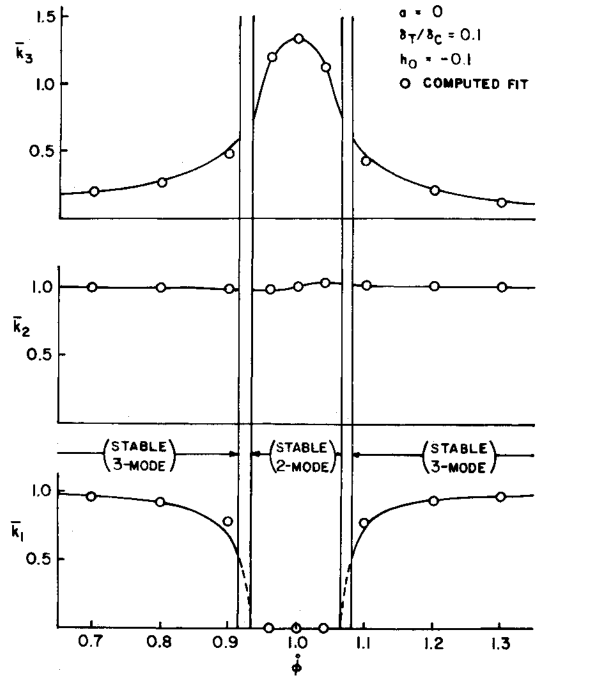

Fig. 6 For $a = 0$ and $\delta_T/\delta_c = 0.1$, a comparison of Eqs. (20-22) with fits of Eq. (19) to generated solutions of Eq. (18).

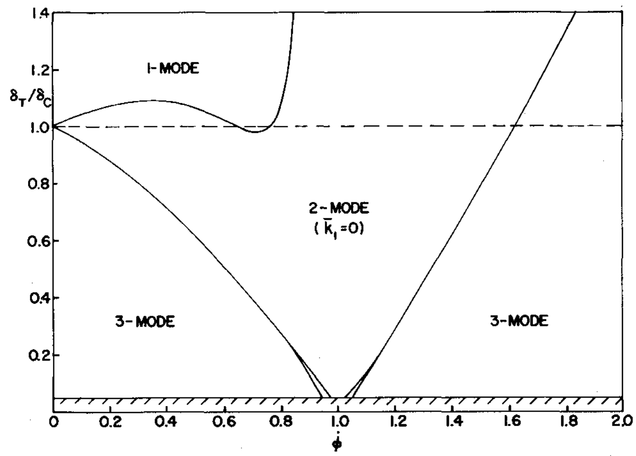

Fig. 8 Regions of stable limit motions for $a = 0$ and $\delta_T/\delta_c > 0.05$.

It is interesting to note that Eqs. (39-41) become considerably simpler for $a = 0$:

$$\bar{k}_1^2 = 1 - (\dot\phi + 1)\bar{k}_3^2 \quad (46)$$

$$\bar{k}_2^2 = 1 + (\dot\phi - 1)\bar{k}_3^2 \quad (47)$$

$$\{(1-\dot\phi^2)^2 + (h_0\dot\phi)^2(1-3\bar{k}_3^2)^2\}\bar{k}_3^2 = (\delta_T/\delta_c)^2 \quad (48)$$

Near resonance, $\dot\phi - 1$ is small and far from resonance $\bar{k}_3^2$ is small so that Eq. (47) gives $\bar{k}_2 \sim 1$ for three-mode motion if $\bar{k}_3$ decays rapidly enough from its resonant value. Since Eq. (31) for two-mode motion also reduces to Eq. (47) when $a = 0$, this approximation is also good for two-mode motion. Finally, Eq. (48) is the same as Eq. (38) and therefore the lower curves of Fig. 1a also apply for the $\bar{k}_3$ component of three-mode motion when $a = 0$.

VIII. Discussion

The general results of the preceding sections give us the strong impression that the behavior of limit motions basically divides into two types—$a > 1$ and $a < 1$. For symmetric missiles, this corresponds to missiles that have circular limit motions and planar limit motions, respectively. A good survey of the behavior of limit motions for slightly asymmetric missiles can be obtained by considering a special case of each type for the full range of spin. These two special cases are $a = 2$ and $a = 0$. The variation of the k_j's for the two special cases is given in Figs. 4 and 6 for $\delta_T/\delta_c = 0.1$ and in Figs. 5 and 7 for $\delta_T/\delta_c = 1.0$.

For $a = 2$ and spin far from resonance, a circular one-mode limit motion occurs with either $\bar{k}_1$ or $\bar{k}_2 = 1$. As spin is varied toward resonance, $\bar{k}_3$ appears and grows, thereby producing a two-mode motion. Eventually the spin reaches a value for lhich $\bar{k}_3$ has grown to 0.707 and $\bar{k}_1$ or $\bar{k}_2$ has become zero. From this value of spin on to resonance, a stable one-mode motion occurs. For $\delta_T/\delta_c = 0.1$ (Fig. 4), there is a small transition region where the two-mode motion is unstable and a more complicated bounded motion must exist. For a large asymmetry ($\delta_T/\delta_c = 1.0$; Fig. 5), the behavior is about the same except that $\bar{k}_3$ is much larger, stable one-mode motion exists over a much larger range of $\dot{\phi}$ and this transition zone does not exist.

For $a=0$ and spin far from resonance, a planar limit motion occurs with $\bar{k}_1 = \bar{k}_2 = 1$, $\bar{k}_3 \sim 0$. As spin is varied toward resonance, $\bar{k}_3$ grows and a three-mode limit motion becomes apparent. Eventually spin reaches a value for which $\bar{k}_3$ has grown to 0.707 and $\bar{k}_1$ has decayed to zero. From this value of spin to resonance, a stable two-mode motion exists. For $\delta_T/\delta_c = 0.1$ (Fig. 6), this motion is simplified by $\bar{k}_2$ remaining quite close to one, although a transition zone of unstable three-mode motion does exist. For larger asymmetry ($\delta_T/\delta_c = 1.0$; Fig. 7), more interesting things happen since $\bar{k}_3$ becomes much larger and $\bar{k}_2$ varies substantially away from unity. Indeed, over an interval of spin, $\bar{k}_2$ is zero and a stable one-mode motion exists!

This quite interesting behavior for $a=0$ is summarized for a range of δ_T/δ_c in Fig. 8. It can be seen that although a pure mode limit motion exists for only a relatively small range of $\dot{\phi}$ when $\delta_T/\delta_c = 1$, for larger asymmetries a pure mode limit motion exists for all spins slightly less than resonance.

As a check on the theory of this paper, a number of numerical integrations of Eq. (18) were performed and fits of Eq. (19) were made to the results. In all cases where stable limit motions were predicted, they were observed; the fitted values of the k_j's are given in Figs. 4-7. The agreement of these results with the quasi-linear theory gives substantial validity to the theory of this paper.

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